

Comparison of invariant functions and metrics

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Abstract. It is shown that all invariant metrics and functions on a bounded \mathcal{C}^2 -smooth domain coincide on an open non-empty subset. The existence of Lempert–Burns–Krantz discs in \mathcal{C}^2 -smooth domains and other possible applications are also discussed.

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1. Introduction. The fundamental Lempert Theorem states that $c_G = l_G$ and $\gamma_G = \kappa_G$ whenever G is convex or a smooth \mathbb{C} -convex domain in \mathbb{C}^n , where c_G is the Carathéodory pseudodistance, l_G denotes the Lempert function, γ_G is the Carathéodory–Reiffen pseudometric, and κ_G is the Kobayashi–Royden pseudometric—see definitions in Section 2.

It is well known that if (d_G) is a contractible family of functions (respectively $(\delta)_G$ a contractible family of pseudometrics), where G goes through the family of all domains in \mathbb{C}^n , then $c_G \leq d_G \leq l_G$ (resp. $\gamma_G \leq \delta_G \leq \kappa_G$). Therefore, the Lempert Theorem may be formulated as follows: on any convex or smooth \mathbb{C} -convex domain of \mathbb{C}^n , all invariant metrics are equal.

This result is surprising as the functions and metrics mentioned above are holomorphic objects (and they are holomorphically invariant) and notions of convexity and \mathbb{C} -convexity are just algebraic (topological) conditions.

There are some results describing properties of invariant metrics on strongly pseudoconvex domains. For example, c_G and k_G (k_G denotes the Kobayashi pseudodistance, that is the biggest pseudodistance less or equal to l_G) are comparable in the sense that for any $\varepsilon > 0$ there is a compact subset $K = K(\varepsilon)$ of a strongly pseudoconvex domain G such that the inequality

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$$c_G(z, w) \leq k_G(z, w) \leq (1 + \varepsilon)c_G(z, w)$$

holds for $(z, w) \in G \times (G \setminus K)$ (see [17]). Another result in this direction is the following one due to Abate (see [1]):

$$\lim_{z \rightarrow \partial G} \frac{c_G(z_0, z)}{-\log \operatorname{dist}(z, \partial G)} = \lim_{z \rightarrow \partial G} \frac{k_G(z_0, z)}{-\log \operatorname{dist}(z, \partial G)} = 1/2,$$

and the limits are locally uniform in the first variable $z_0 \in G$ (for some other results see [15] and references there).

Nevertheless, it is well known that invariant functions (metrics) are usually quite different even on smooth strongly pseudoconvex domains (and even when $n = 1$) (see e.g. Section 5).

Our main result of this short note states that if G is a \mathcal{C}^2 -smooth bounded domain in \mathbb{C}^n (with no additional assumptions such a pseudoconvexity), where $n \geq 2$, then all invariant function (resp. metrics) are equal on some open subset of $G \times G$ (resp. $G \times \mathbb{C}^n$). The idea of the proof relies upon applying a counterpart of the Pinchuck scaling method and making use of stationary mappings which were used by Lempert in proving his deep result.

Then we show that modifying this method, we may obtain the existence of Lempert–Burns–Krantz discs in \mathcal{C}^2 smooth domains (see [11] for this definition). We end the paper with some technical lemmas (which follow simply from well known results and estimates).

2. Notation and definitions. We start with some notation, and we recall basic definitions appearing in the theory of invariant function and metrics. For a comprehensive monograph on this subject, we refer the reader to [9].

Throughout the paper \mathbb{D} denotes the unit disc in the complex plain and \mathbb{T} denotes the circle $\partial\mathbb{D}$. By $\langle z, w \rangle$ we denote the complex inner product on \mathbb{C}^n , and $z \bullet w$ is the dot product, i.e. $z \bullet w = \langle z, \bar{w} \rangle$, $z, w \in \mathbb{C}^n$.

Let $D \subset \mathbb{C}^n$ be a domain, and let $z, w \in D$, $v \in \mathbb{C}^n$. The *Lempert function* is defined as

$$l_D(z, w) := \inf \{ p(0, \xi) : \xi \in [0, 1) \text{ and } \exists f \in \mathcal{O}(\mathbb{D}, D) : f(0) = z, f(\xi) = w \}. \quad (1)$$

The *Kobayashi–Royden (pseudo)metric* we define as

$$\kappa_D(z; v) := \inf \{ \lambda^{-1} : \lambda > 0 \text{ and } \exists f \in \mathcal{O}(\mathbb{D}, D) : f(0) = z, f'(0) = \lambda v \}. \quad (2)$$

It is well known that generally l_D does not satisfy a triangle inequality. Therefore, it is natural to consider the so-called *Kobayashi (pseudo)distance* given by the formula

$$k_D(z, w) := \sup \{ d_D(z, w) : (d_D) \text{ is a family of holomorphically invariant pseudodistances less than or equal to } l_D \}.$$

Clearly,

$$k_D(z, w) = \inf \left\{ \sum_{j=1}^N l_D(z_{j-1}, z_j) : N \in \mathbb{N}, z_1, \dots, z_N \in D, z_0 = z, z_N = w \right\}.$$

The next objects we are dealing with are the *Carathéodory (pseudo)distance*

$$c_D(z, w) := \sup\{p(F(z), F(w)) : F \in \mathcal{O}(D, \mathbb{D})\}$$

and the *Carathéodory-Reiffen (pseudo)metric*

$$\gamma_D(z; v) := \sup\{|F'(z)v| : F \in \mathcal{O}(D, \mathbb{D}), F(z) = 0\}.$$

Recall that a holomorphic mapping $f : \mathbb{D} \rightarrow D$ is said to be a *complex geodesic* if for any $z, w \in f(\mathbb{D})$ there are $\zeta, \xi \in \mathbb{D}$ such that $f(\zeta) = z$, $f(\xi) = w$, and $c_D(z, w) = p(\zeta, \xi)$ (resp. there are $\lambda_0 \in \mathbb{D}$, $\alpha_0 \in \mathbb{C}$ such that $f(\lambda_0) = z$, $X = \alpha_0 f'(\lambda_0)$ and $\gamma_D(z, X) = \gamma_D(\lambda_0, \alpha_0)$).

Moreover, $f : \mathbb{D} \rightarrow D$ is called an *extremal* mapping if $k_D(z, w) = p(\zeta, \xi)$ for some $\zeta, \xi \in \mathbb{D}$ such that $f(\zeta) = z$ and $f(\xi) = w$ (resp. there are $\lambda_0 \in \mathbb{D}$, $\alpha_0 \in \mathbb{C}$ such that $f(\lambda_0) = z$, $X = \alpha_0 f'(\lambda_0)$ and $\kappa_D(z, X) = \gamma_D(\lambda_0, \alpha_0)$).

Lempert introduced the concept of stationary map. This idea was originally derived by solving the Euler–Lagrange equations for the extremal problem. Let D be a C^2 -smooth, bounded domain. Recall that $f : \mathbb{D} \rightarrow D$ is a *stationary mapping* if

- (1) f extends to a $C^{1/2}$ -smooth mapping on $\overline{\mathbb{D}}$;
- (2) $f(\mathbb{T}) \subset \partial D$;
- (3) there exists a $C^{1/2}$ -smooth function $\rho : \mathbb{T} \rightarrow \mathbb{R}_{>0}$ such that the mapping $\mathbb{T} \ni \zeta \mapsto \zeta \rho(\zeta) \overline{\nu_D(f(\zeta))} \in \mathbb{C}^n$ extends to a mapping $\tilde{f} \in \mathcal{O}(\mathbb{D}) \cap C^{1/2}(\overline{\mathbb{D}})$ (we call \tilde{f} a *dual map* to f).

Note that if D is additionally convex and f is a stationary mapping in D , then $\operatorname{Re}\langle z - f(\zeta), \nu_D(f(\zeta)) \rangle < 0$ for any $z \in D$ and $\zeta \in \mathbb{T}$. Therefore, for any $z \in D$ the equation $(z - f(\cdot)) \bullet \tilde{f}(\cdot) = 0$ has exactly one solution $F(z)$ in \mathbb{D} . One may check that F is a left inverse for f , i.e., $F \circ f = \operatorname{id}$.

3. Equality of invariant metrics and functions. As mentioned in the Introduction, our main result is the following

Theorem 1. *Let $D \subset \mathbb{C}^n$, $n \geq 2$, be a domain with C^2 boundary.*

Then there is a non-empty and open subset U of $D \times D$ such that

$$c_D(z, w) = l_D(z, w) \quad \text{for } (z, w) \in U.$$

Similarly, there is a non-empty and open subset V of $D \times \mathbb{C}^n$ such that

$$\kappa_D(z, X) = \gamma_D(z, X) \quad \text{for } (z, X) \in V.$$

Note that the assumption $n \geq 2$ is important. Actually, it is well known that

$$c_A(z, w) < k_A(z, w), \quad (z, w) \in A \times A,$$

where A is an annulus in the complex plane, that is $A := \{z \in \mathbb{C} : r < |z| < 1\}$, where $r < 1$.

Similarly, the product property of the invariant metrics and pseudodistances ensures that the assumption about the smoothness is important as well (consider the product of annuli).

We would like to point out that Theorem 1 remains true if C^2 -smoothness will be replaced with $C^{1,1}$ -smoothness. The differences in the proofs between

these two cases are only technical, so for simplicity we shall consider only the C^2 case.

Lemma 1. *Let $D \subset \mathbb{C}^n$, $n \geq 2$, be a domain. Assume that $a \in \partial D$ is such that ∂D is C^2 and strongly convex in a neighborhood of a . Then for any neighborhood V_0 of a , there is a non-empty and open $U \subset (V_0 \cap D) \times (V_0 \cap D)$ with the following property:*

For any $(z, w) \in U$ there is a stationary mapping f of $D \cap V_0$ passing through (z, w) such that $f(\mathbb{T}) \subset \partial D$.

Proof. Let r be a C^2 defining function in a neighborhood of a . The problem we are dealing with has a local character, and a is a point of strong convexity. Therefore, analyzing the Taylor series, one may simply see that replacing r with $r \circ \Psi$, where Ψ is a local biholomorphism near a , we may assume that $a = (0, \dots, 0, 1)$ and a defining function of D near a is of the form $r(z) = -1 + \|z\|^2 + h(z - a)$, where h is C^2 smooth in a neighborhood of 0 and

$$h(z) = o(\|z\|^2), \quad \text{as } z \rightarrow 0. \quad (3)$$

Of course, the similar holds for partial derivatives of h of first order, i.e.

$$D^\alpha h(z) = o(\|z\|), \quad \text{as } z \rightarrow 0, \quad \text{for any } \alpha \in \mathbb{N}^n, \quad |\alpha| = 1. \quad (4)$$

In particular, $D^\alpha h(0) = 0$ for any $\alpha \in \mathbb{N}_0^n$ such that $|\alpha| = 2$.

Similarly, as in [13] we consider the mappings¹

$$A_t(z) := \left((1 - t^2)^{1/2} \frac{z'}{1 + tz_n}, \frac{z_n + t}{1 + tz_n} \right), \quad z = (z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{D}, \quad t \in (0, 1).$$

Note that A_t is an automorphism of \mathbb{B}_n . Let

$$r_t(z) := \frac{|1 + tz_n|^2}{1 - t^2} r(A_t(z)), \quad t \in (0, 1). \quad (5)$$

After some elementary calculations, we infer that

$$\begin{aligned} r_t(z) &= -1 + \|z\|^2 + \frac{|1 + tz_n|^2}{1 - t^2} h(A_t(z) - a) \\ &= -1 + \|z\|^2 + \frac{|1 + tz_n|^2}{1 - t^2} h \left((1 - t^2)^{1/2} \frac{z'}{1 + tz_n}, (1 - t) \frac{z_n - 1}{1 + tz_n} \right). \end{aligned} \quad (6)$$

Take any increasing sequence $\{t_\mu\}$ converging to 1. Put $U_0 := \{z \in \mathbb{C}^n : \operatorname{Re} z_n > -1/2\}$ and define $\rho(z) := -1 + \|z\|^2$, $z \in \mathbb{C}^n$.

Observe that it follows from (6) that $r_{t_\mu}|_{U_0}$ converges to $\rho|_{U_0}$ in the $\mathcal{C}^2(U_0)$ topology. Actually, the local uniform convergence of the functions r_{t_μ} follows simply from (3). Similarly, making use of (4), one may deduce the local uniform convergence of partial derivatives of the first order. The local uniform convergence of the partial derivatives of r_{t_μ} is a consequence of the continuity of the second-order partial derivatives of h .

¹ It should be noticed that here we use a version of the Pinchuck scaling method.

Let χ be a \mathcal{C}^∞ smooth function on \mathbb{R} such that

- $\chi \equiv 0$ on $(-\infty, -1/2]$;
- $\chi \equiv 1$ on $[-1/4, \infty)$;
- χ is increasing.

Let us define

$$\tilde{\rho}_\mu(z) := \begin{cases} r_{t_\mu}(z), & -1/2 \leq \operatorname{Re} z_n, \\ \chi(\operatorname{Re} z_n) r_\mu(z) + (1 - \chi(\operatorname{Re} z_n)) \rho(z), & -1/4 \leq \operatorname{Re} z_n < -1/2, \\ \rho(z), & \operatorname{Re} z_n < -1/2. \end{cases} \quad (7)$$

Since r_{t_μ} converges to ρ in the \mathcal{C}^2 topology on U_0 , we easily infer that $\tilde{\rho}_\mu$ converges to ρ in the \mathcal{C}^2 topology on \mathbb{C}^n .

In particular, $(\tilde{\rho}_\mu)$ converges locally uniformly to ρ . Let (ε_μ) be a sequence of positive numbers converging to 0 such that $3\varepsilon_{\mu+1} < \varepsilon_\mu$. There is a subsequence (ρ_{s_μ}) such that $\sup_{\mathbb{B}_n} |\rho_{s_\mu} - \rho| < \varepsilon_\mu$. Therefore, $\rho_\mu := \tilde{\rho}_{s_\mu} + 2\varepsilon_\mu$ restricted to $\overline{\mathbb{B}_n}$ is strictly decreasing.

Let D_μ be the connected component of $\{\rho_\mu < 0\}$ containing 0. Clearly D_μ are strongly convex provided that μ is big enough, as ρ_ν converges to ρ in the \mathcal{C}^2 topology. Since D_μ increase to \mathbb{B} , we find that the sequence l_{D_μ} decreases $l_{\mathbb{B}_n}$. Since ρ_μ converges to ρ in the \mathcal{C}^2 topology, we easily deduce that there is a uniform $c > 0$ such that any geodesic in D_μ such that $\operatorname{dist}(f(0), \partial D_\mu) > 1/c$ is $\mathcal{C}^{1/2}$ continuous, and its $\mathcal{C}^{1/2}$ norm depends only on μ providing that μ is sufficiently large.

Now we proceed as follows. Let $V \subset \mathbb{B}_n \times \mathbb{B}_n$ be open and such that any geodesic passing through $(z, w) \in \overline{V}$ lies entirely in $U_1 = \{\operatorname{Re} z_n > -1/4\}$. Let W be non-empty, open, and relatively compact in V .

It follows that for any $(z, w) \in W$ there is a geodesic f_μ in D_μ such that $f_\mu(0) = z$ and $f_\mu(\sigma_\mu) = w$ for some $\sigma_\mu > 0$. Passing, if necessary, to a subsequence we may assume that f_μ converges to a mapping $f_0 : \mathbb{D} \rightarrow \overline{\mathbb{B}_n}$. Since $f_0(0) = z \in \mathbb{B}_n$ we get that $f_0(\mathbb{D}) \subset \mathbb{B}_n$. Moreover, the statement of the Lempert Theorem holds on D_μ that is $c_{D_\mu} = l_{D_\mu}$, hence we may easily see that f_0 is a complex geodesic in \mathbb{B}_n passing through (z, w) . Then uniqueness, uniform convergence and $\mathcal{C}^{1/2}$ uniform continuity implies that $f_\mu(\overline{\mathbb{D}})$ lies entirely in $\{\operatorname{Re} z_n > -1/2\}$ provided that $\mu = \mu(z, w)$ is big enough—see Lemma 9 below.

Thus, a standard Baire argument implies the existence of an open non-empty subset W of V and, a natural ν_1 such that for any $(z, w) \in W$ and $\nu \geq \nu_1$, a geodesic of D_n passing through (z, w) (let us denote it by $f_{\nu, (z, w)}$) lies entirely in W . Actually, it suffices to apply the Baire theorem to the family $\{G_\mu\}$, where $G_\mu := \{(z, w) \in \bar{V} : f_{\nu, (z, w)}(\overline{\mathbb{D}}) \subset \bar{U}_1, \nu \geq \mu\}$.

Observe that $g_\nu := A_{t_{s_\nu}} \circ f_\nu$ is a stationary mapping of D . Since g_ν maps \mathbb{D} onto arbitrarily small neighborhoods of a provided that ν is sufficiently big, we immediately get the assertion. \square

Proof of Theorem 1. Losing no generality let us assume that $0 \in D$. Fix a point a in the topological boundary of D whose distance from 0 is the biggest.

Then a is a point of strict convexity of D . Let U' be an open and convex neighborhood of a such that $D \cap U'$ is convex. Moreover

$$\operatorname{Re}\langle z - a, \nu_D(a) \rangle < 0 \quad \text{for any } z \in D. \quad (8)$$

Let $U'' \subset\subset U'$ be a neighborhood of a with the following property

(†) for any $\zeta \in \partial D \cap U''$ and any $z \in D \setminus U'$, one has the inequality $\operatorname{Re}\langle z - \zeta, \nu_D(\zeta) \rangle \leq 0$.

Making use of Lemma 1, we get an open set U in $D \times D$ such that for any $(z, w) \in U$ there is a weak stationary mapping of $D \cap U''$ passing through (z, w) and entirely contained in $D \cap U''$. In particular,

$$\operatorname{Re}\langle z - f(\eta), \nu_D(f(\eta)) \rangle < 0, \quad \zeta \in \mathbb{T} \quad (9)$$

for any $z \in U''$. Since $D \cap U'$ is convex, we find that (9) holds for any $z \in U'$.

Making use of (†), we infer that (9) holds on the whole D .

From this we easily deduce that f has a left inverse $F : D \rightarrow \mathbb{D}$, hence f is a complex geodesic (actually, $F(z)$ may be obtained as a unique solution of the equation $\langle z - f(\eta), \tilde{f}(\eta) \rangle = 0$ with unknown $\eta \in \mathbb{D}$, where \tilde{f} is a dual map of f). \square

Let D be a \mathcal{C}^2 smooth strongly pseudoconvex domain in \mathbb{C}^n , and let $a \in \partial D$. The theorem of Fornaess (see [7], Proposition 1) gives a neighborhood B of a , a strictly convex domain C of \mathbb{C}^n , a mapping $\Phi : D \rightarrow \mathbb{C}^n$ extending holomorphically to a neighborhood of D such that $\Phi(D) \subset C$, $\Phi(B \setminus \overline{D}) \subset \mathbb{C}^n \setminus \overline{C}$, $\Phi^{-1}(\Phi(B)) = B$ and the restriction $\Phi|_B : B \rightarrow \Phi(B)$ is biholomorphic (see also [5] where this result is superseded). It follows from the reasoning presented above that we may construct a complex geodesic f in C lying entirely in $\Phi(B)$. Then $(\Phi|_B)^{-1} \circ f$ is a complex geodesic in G ($F \circ \Phi$ is its left inverse, where F is a left inverse of f in G).

Using this standard reasoning, we easily get the following:

Theorem 2. *Let D be a \mathcal{C}^2 -smooth strongly pseudoconvex domain in \mathbb{C}^n . Then for any $z_0 \in \partial D$ and any neighborhood of U of z_0 , there is a non-empty and open subset V of $U \times U$ such that $c_D = \kappa_D$ on V .*

4. Lempert–Burns–Krantz discs in \mathcal{C}^2 -smooth domains. The main goal of this section is to apply the method presented above in order to show the existence of the so called Lempert–Burns–Krantz discs in \mathcal{C}^2 -smooth strongly pseudoconvex domains. Recall that a Lempert–Burns–Krantz disc for points $a \in \partial D$ and $b \in D$, where D is a domain of \mathbb{C}^n , is a geodesic f of D , continuous up to \mathbb{D} such that $f(1) = a$ and $b \in f(\mathbb{D})$.

More precisely, we shall show the following

Theorem 3. *Let Ω be a \mathcal{C}^2 -smoothly bounded strongly pseudoconvex in \mathbb{C}^n . Fix $p \in \partial\Omega$. Then there is a non-empty and open subset V of Ω such that for any $q \in V$ there exists an $f : \mathbb{D} \rightarrow \Omega$ such that*

- f is a complex geodesic in Ω ,
- $f \in \mathcal{C}^{1/2}(\mathbb{D})$ (in particular, f extends continuously to $\overline{\mathbb{D}}$),
- $q \in f(\mathbb{D})$ and $f(1) = p$.

Moreover, V may be arbitrarily close to p .

Remark 2. Note that using the argument similar to the one used in the proof of Theorem 1, one may show that any \mathcal{C}^2 -smooth domain admits Lempert–Burns–Krantz discs.

In the case when the domain D is \mathcal{C}^6 -smooth, Theorem 3 was proved by Lempert in [13] and formulated in the form above in [2]. It should be noted that the Lempert method may be modified so that it works in the case of $\mathcal{C}^{2+\epsilon}$ -smooth strictly pseudoconvex domains (see [14] for details). However, it cannot be applied in the \mathcal{C}^2 -smooth case (more precisely, the crucial step of Lempert’s arguments relied upon the implicit function theorem to the mapping which is not differentiable assuming only \mathcal{C}^2 -smoothness).

The proof presented here is just a modification of the argument used in Section 1. Note that we cannot use here a Baire-type argument and more subtle reasoning is necessary (we shall make use of estimates which are postponed to Section 7).

Proof. As mentioned above the proof is just a slight modification of the proof of Theorem 1, so we shall follow it. So we may assume that $r(z) = -1 + \|z\|^2 + h(z - a)$, where $a = (0, \dots, 0, 1) \in \mathbb{C}^n$ and h is \mathcal{C}^2 -smooth in a neighborhood of 0, $h(z) = o(\|z\|^2)$, as $z \rightarrow 0$. Let r_t be given by (5) and, similarly as in (7), put

$$\rho_t(z) := \begin{cases} r_t(z), & -1/2 \leq \operatorname{Re} z_n, \\ \chi(\operatorname{Re} z_n)r_t(z) + (1 - \chi(\operatorname{Re} z_n))\rho(z), & -1/4 \leq \operatorname{Re} z_n < -1/2, \\ \rho(z), & \operatorname{Re} z_n < -1/2, \end{cases} \quad (10)$$

where $\rho(z) = -1 + \|z\|^2$, $z \in \mathbb{C}^n$. First observe that $D_t := \{z \in \mathbb{C}^n : \rho_t(z) < 0\}$ is connected, strongly pseudoconvex, and a lies in its boundary provided that t is close enough to 1. Take any open sets U and V such that U is relatively compact in \mathbb{B}_n , $a \in V$, and any geodesic of \mathbb{B}_n passing through points $z \in U$ and $w \in V$ lies entirely in $\{\operatorname{Re} z_n > 0\}$. Let t be big enough (how big enough will be defined later). Fix $z \in U$ and take any sequence $(a_\nu) \subset D_t$ converging to a . Since D_t is strictly convex, we get that there is a complex geodesic f_ν of D_t such that $f_\nu(0) = z$ and $f_\nu(\alpha_\nu) = a_\nu$ for some $\alpha_\nu \in (0, 1)$ ($U \subset\subset D_t$ whenever t is close to 1). Clearly α_ν tends to 1. Since f_ν are uniformly $\mathcal{C}^{1/2}$ -continuous, we may find a subsequence of f_ν converging to a complex geodesic $f_z \in \mathcal{C}^{1/2}(\mathbb{D})$ of D_t such that $f_z(0) = z$ and $f_z(1) = a$ (see also [4], where the similar argument was used). Now, it follows from Lemma 12 that any complex geodesic g of \mathbb{B}_n for the pair $(z, f'_z(0))$ is close (in the sup-norm) to f_z . In particular, $g(1) \in V$ (if t is big enough). Clearly g is a complex geodesic for $z \in U$ and $g(1) \in V$. Thus the image $g(\mathbb{D})$ lies in $\{\operatorname{Re} z_n > 0\}$. Therefore f_z lies entirely in $\{\operatorname{Re} z_n > -1/2\}$ (provided that t is big enough).

So fixing t sufficiently close to 1, we easily verify that $\tilde{U} := A_t(U)$ may be arbitrarily close to a (as $A_t(\cdot) \rightarrow a$ uniformly on compact subsets of $\{\operatorname{Re} z_n > -1\}$). Since stationary mappings are invariant under biholomorphisms, we infer that $g_z := A_t \circ f_z$ is an E-mapping in D passing through $A_t(z)$ and a , whose

image is contained in \tilde{U} . Since U may be arbitrary small, we may apply [7]. Since E-mappings are geodesics on convex domains, we easily find that g_z is a complex geodesic. \square

Remark 3. It is well known that a geodesic f of \mathcal{C}^2 -smooth bounded convex domain is \mathcal{C}^α for any $\alpha < 1$. Note that if we knew that there is a left inverse for f of class \mathcal{C}^α , we would be able to formulate the Burns–Krantz theorem for D .

Of course, if \tilde{f} were of \mathcal{C}^1 class, where \tilde{f} is the dual map to f , then making use of the implicit function theorem and the equality $f' \bullet \tilde{f} = 1$, we would easily find a \mathcal{C}^1 -smooth left inverse. Note that such a statement was claimed in [16].

5. Examples. As mentioned above we have the following

Example 4. Let $0 \leq r_- < r_+ < \infty$ and $A := \{z \in \mathbb{C} : r_- < |z| < r_+\}/$. Then

$$c_A(z, w) < l_A(z, w) = k_A(z, w)$$

for any $(z, w) \in A \times A$.

Similarly, making use of the product property of invariant metrics, we get that

$$c_{A^n}(z, w) < l_{A^n}(z, w) = k_{A^n}(z, w), \quad z, w \in A^n.$$

Example 5. Let $D_\alpha = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, |z_2| < 1, |z_1 z_2| < \alpha\}$, where $\alpha \in (0, 1]$. It may be shown (see e.g. [9, 10]) that any extremal mapping of D_α intersects one of the axes. Therefore, putting $D := D_\alpha \cap \mathbb{C}_*^2$ we get

$$c_D(z, w) < l_D(z, w) = k_D(z, w), \quad (z, w) \in D^2.$$

6. Some other applications. First observe that the geodesics constructed in Lemma 1 are unique. More precisely:

Proposition 6. *Under the assumptions of Theorem 1, there is a non empty and open subset U of $D \times D$ with the following property: for any $(z, w) \in U$ there is exactly one complex geodesic in D passing through z and w .*

Proof. We keep the notation from the proof of Theorem 1, we proceed as Lempert in [13]. First recall that with any f geodesic f of D contained entirely in U'' , we may associate its dual map \tilde{f} . This means that $\tilde{f} \in \mathcal{O}(D, \mathbb{C}^n) \cap \mathcal{C}^{1/2}(\mathbb{D})$ and $\tilde{f}(\zeta) = \zeta p(\zeta) \overline{\nu_D(f(\zeta))}$, $\zeta \in \mathbb{T}$ for some positive, $\mathcal{C}^{1/2}$ -smooth function p .

The equation (\dagger) means that for any $z \in D$:

$$\operatorname{Re}(z - f(\zeta)) \bullet \frac{\tilde{f}(\zeta)}{\zeta} < 0, \quad \zeta \in \mathbb{T}.$$

In particular, for any $z \in D$ the equation $(z - f(\zeta)) \bullet \tilde{f}(\zeta) = 0$ has exactly one solution $\zeta \in \mathbb{D}$ denoted by $F(z)$. Clearly, F is a left inverse for f .

To prove the uniqueness, suppose that $f_1 : \mathbb{D} \rightarrow D$ is a complex geodesic in D passing through $(z, w) \in U$. We have to show that f_1 is equal to f . Since D is bounded, non-tangential limits f_1^* exist almost everywhere on \mathbb{T} . Clearly,

$F \circ f_1 = F \circ f = \text{id}$. This in particular means that $\langle f_1^*(\zeta) - f(\zeta), \nu_D(f(\zeta)) \rangle = 0$ for $\zeta \in \mathbb{T}$. A strong convexity of D in a neighborhood of a implies that $f_1^* = f$ a.e. on \mathbb{T} , so the identity principle finishes the proof. \square

The result presented above may be used in proving that a smooth domain is not biholomorphic (or there is no proper holomorphic mapping) with a domain whose geodesics are not uniquely determined. Below we present such an application.

Lemma 7. *Let $D = D_1 \times D_2 \subset \mathbb{C}^{n_1} \times \mathbb{C}^{n_2} = \mathbb{C}^n$ be a domain, $n_1, n_2 > 0$. Then the set of points (z, w) in $D \times D$ possessing a unique complex geodesic has empty interior.*

Proof. This is a simple consequence of the product formula

$$c_D((z_1, z_2), (w_1, w_2)) = \max\{c_{D_1}(z_1, w_1), c_{D_2}(z_2, w_2)\}$$

(see e.g. [9]) and the fact that the sublevels $\{z \in D_i : c_D(z_i, w) = \alpha\}$ have empty interior for any $w \in D_i$ and $\alpha > 0$ (which clearly follows from the fact that non-constants holomorphic functions are open). \square

This gives the following corollary (see also [3, 8]):

Example 8. Let D be a bounded and smooth. Then it is not biholomorphic with a Cartesian product of non-empty domains.

An interesting application of the above result is that there is no domain fulfilling the assumptions of Theorem 13 in [16].

7. Technical lemmas. Recall that a complex geodesic of a strictly pseudoconvex domain is $\mathcal{C}^{1/2}$ -continuous and speaking very generally its $\mathcal{C}^{1/2}$ depends only on the curvatures of the topological boundary of a domain, its diameter and the distance between $f(0)$ and ∂D . See [13] for details (the results presented there are formulated for extremals in strictly pseudoconvex domains, but their proofs work for complex geodesics in arbitrary strictly pseudoconvex domains).

Lemma 9. *Assume that $D \subset \subset \mathbb{B}_n$ is a bounded \mathcal{C}^2 smooth strongly pseudoconvex domain of \mathbb{C}^n , $0 \in D$. Let $r \in \mathcal{C}^2(\mathbb{B}_n)$ be a defining function of D such that $\mathcal{L}r(a, X) \geq \alpha \|X\|^2$, $a \in \partial D$, $X \in \mathbb{C}^n$ for some $\alpha > 0$. Let $(r_\mu) \subset \mathcal{C}^2(\mathbb{B}_n)$ be a sequence converging to r in a \mathcal{C}^2 topology on \mathbb{B}_n . By D_μ we denote the connected component of $\{z \in \mathbb{B}_n : r_\mu(z) < 0\}$ containing the origin. Fix a compact subset K of D .*

Then every l_{D_μ} -geodesic f such that $f(0) \in K$ is $\mathcal{C}^{1/2}$ continuous, and its $\mathcal{C}^{1/2}$ may be estimated by a constant independent of μ for μ big enough.

Sketch of the proof of Lemma 9. One may easily show that there is $c > 0$ such that D and D_μ are in $\mathcal{D}(c)$ for $\mu \gg 1$, where $\mathcal{D}(c)$ is the family defined in [14]. Thus it suffices to observe that Propositions 7 and 8 of [14] work for \mathcal{C}^2 -smooth domains when we replace the assumption of being an E -mapping by the assumption of being a geodesic (the proofs given there may be taken over verbatim). \square

Remark 10. It is worth mentioning that using the argument of [4], which also works in a \mathcal{C}^2 -smooth domain strictly pseudoconvex domain and keeping the notation from the lemma above, one may get that any l_{D_μ} -geodesic such that has up to a composition with a Möbius map $\mathcal{C}^{1/4}$ norm bounded by a constant independent of μ .

Note that the assumptions of the lemma presented above imply that $\{r_\mu < 0\}$ is connected for $\mu \gg 1$. Moreover, using some simple calculus, one may show that δ_{D_μ} converges to δ_D in the \mathcal{C}^2 topology, where δ_G denotes the signed distance to the bounded domain $G \subset \mathbb{C}^n$, i.e.

$$\delta_G(z) := \begin{cases} -\text{dist}(z, \partial G), & \text{if } z \in \overline{G}, \\ \text{dist}(z, \partial G), & \text{if } z \notin \overline{G}. \end{cases}$$

Remark 11. We would like to recall once again that the lemmas above may also be formulated and proved in the $\mathcal{C}^{1,1}$ case (then the condition on curvatures should be naturally replaced by an exterior and interior ball condition).

If D is a bounded \mathcal{C}^2 -smooth strictly convex domain, then for every $z \in D$ and $X \in \mathbb{S}_{n-1} := \partial \mathbb{B}_n$, there is a unique \mathcal{C}^α -smooth, $\alpha < 0$, geodesic in D for (z, X) , denoted by $f_{z,X}$, such that $f_{z,X}(0) = z$ and $f'_{z,X}(0) = \lambda_{z,X}X$ for some $\lambda_{z,X} > 0$.

Lemma 12. *Let D be a \mathcal{C}^2 -smooth strictly convex domain, and let r_0 be its defining function given on a neighborhood V of \overline{D} . Let K be a compact subset of D . For $r \in \mathcal{C}^2(V)$ sufficiently close to r_0 in the \mathcal{C}^2 topology on V , let D_r denote the connected component of $\{x \in V : r(x) < 0\}$ containing 0 (note that D_r is strictly convex providing that r is sufficiently close to r_0). Let $f^r_{z,X}$ and $f_{z,X}$ be complex geodesics of D_r and D respectively such that $f^r_{z,X}(0) = f_{z,X}(0) = z$, $(f^r_{z,X})'(0) = \lambda^r_{z,X}X$, and $f'_{z,X}(0) = \lambda_{z,X}X$ for some $\lambda^r_{z,X}, \lambda_{z,X} > 0$, where $z \in K$ and $X \in \mathbb{S}_{n-1}$.*

Then $\|f^r_{z,X} - f_{z,X}\|_\infty \rightarrow 0$ as $r \rightarrow r_0$ in the \mathcal{C}^2 topology on V .

Proof. This is just a simple consequence of the uniqueness of complex geodesics in D , Lemma 9, and a standard compactness argument. \square

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